New vortex solution in $SU(3)$ gauge-Higgs theory

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Following a brief review of known vortex solutions in $SU(N)$ gauge-adjoint Higgs theories we show the existence of a new ''minimal'' vortex solution in $SU(3)$ gauge theory with two adjoint Higgs bosons. At a critical coupling the vortex satisfies Bogomol'nyi type, first order, field equations. The exact value of the vortex energy (per unit length) is found in terms of the topological charge at the critical coupling.

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I. INTRODUCTION

Classical solutions of non-Abelian gauge theories have played an important role in a variety of contexts [1]. Classical solutions in Higgs theories may play an important role in cosmology [2]. They may also be relevant in models of confinement [3]. Different classical objects may affect cosmology, symmetry breaking, etc., in different ways. Therefore, it is of considerable importance to find all classical solutions and investigate their properties.

Vortex solutions are solitons in 2+1 dimensions and are stringlike extended objects in 3+1 dimensions. In 3+1 dimensions they have infinite energy (the energy per unit length is finite) but condensed vortices contribute a finite amount to the free energy per unit volume. Non-Abelian vortex configurations were discussed in [4–6]; explicit vortex solutions were first found in Ref. [7]. The existence of non-Abelian vortices is the consequence of nontrivial topological classes in the mapping $S_1 \to SU(N)/Z_N$. The homotopy group of this mapping is Z_N, implying the existence of $N-1$ distinct stable vortices. As the symmetry, classifying vortices, is the center of the gauge group $SU(N)$, one needs to introduce Higgs fields that break $SU(N)$ symmetry, but not the center Z_N. The smallest representation for the Higgs fields, such that they commute with the center, is the adjoint representation. Therefore, one needs to use one or more adjoint Higgs bosons to break the symmetry. Symmetry breaking induced by a single adjoint Higgs boson is not complete. The adjoint Higgs boson, when diagonalized, commutes with the ''diagonal'' generators, the elements of the Cartan subgroup, $[U(1)]^{N-1}$. The relevant classical objects in such a theory are 't Hooft–Polyakov monopoles [8,9]. Thus, at least two adjoint Higgs bosons are needed to break the symmetry down to its center.

Vortex solutions found in [7] correspond to $SU(N)$ adjoint Higgs theories with N Higgs bosons. In fact, one would think that a ''minimal'' solution could be found with only two Higgs bosons. The first Higgs boson breaks the symmetry down to the maximal Abelian subgroup and then another Higgs boson, which is kept non-parallel with the first one, can break all the remaining continuous symmetries. The purpose of this paper is to show that vortex solutions in $SU(3)$ gauge theory with two adjoint Higgs bosons exist and to study the properties of these solutions.

The equations of motion in Abelian [10,11] and non-Abelian [12] vortex model were shown to reduce to linear, Bogomol'nyi equations at critical values of the coupling constant. This phenomenon was shown to be related to the increase of an underlying supersymmetry of the model [13,14]. The equations of motions we obtain for the $SU(3)$ Higgs theory also linearize at critical couplings. The linearization of the field equations may also be related to the increase of supersymmetry.

In the next section we will briefly review the solutions of field equations for $SU(3)$ theory offered in Ref. [7]. In Sec. III we will present our two-Higgs-boson model and the ansatz for solving the equations of motion. In Sec. IV we will discuss the critical coupling and the Bogomol'nyi equations, followed by a concluding section.

II. VORTEX SOLUTIONS IN *SU(N)* GAUGE THEORY WITH *N* HIGGS BOSONS

As usual in discussing time-independent classical solutions we will consider the Hamiltonian, the negative of the Lagrangian in the absence of time derivative. The Hamiltonian for a cylindrically symmetric solution is of the form

$$H=\int d^2x\left[\frac{1}{4}G_{\mu\nu}^2+\frac{1}{2}\sum_{A=1}^{N}(D_\mu\Phi^{(A)})^2+V(\Phi^{(A)})\right]. \quad (1)$$

Here

$$A_\mu=A_\mu^a t^a,\quad a=1,2,\dots,N^2-1,$$

$$D_\mu=\partial_\mu+ie[A_\mu,\ ],$$

$$G_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+ie[A_\mu,A_\nu] \quad (2)$$

where t^a are the $SU(N)$ generators. We are considering N Higgs scalars $\Phi^{(A)}$ in the adjoint representation and the potential $V[\Phi]$ chosen so as to ensure complete symmetry breaking.

Vortex solutions to the equations of motion associated with Hamiltonian (1) have been found in [7] by making an ansatz that ensures non-trivial topology and maximum symmetry breaking. Since the scalars are in the adjoint representation, the center Z_N of $SU(N)$ is the surviving symmetry subgroup. Then, the relevant homotopy group for classifying topologically inequivalent configurations is non-trivial, $\pi_1(SU(N)/Z_N)=Z_N$. One then has $N-1$ topologically non-trivial inequivalent possible solutions which can be associated with gauge group elements U_n ($n=1,2,\dots,N-1$ labeling the homotopy classes). If we call ϕ the azimuthal angle in a plane perpendicular to the vortex, then $U_n(\phi)$ should satisfy, when a turn around a closed contour is made,

$$U_n(\phi+2\pi)=\exp\left(i\frac{2\pi(n+Nk)}{N}\right)U_n(\phi),\quad n=1,2,\dots,N-1,\quad k\in Z. \tag{3}$$

Condition (3) can be realized by writing

$$U_n(\phi)=\mathrm{diag}\left[\exp\left(i(n+Nk)\frac{\phi}{N}\right),\dots,\times\exp\left(i(n+Nk)\frac{\phi}{N}\right),\exp\left(-i\phi\frac{N-1}{N}(n+Nk)\right)\right]. \tag{4}$$

Then, one can construct a gauge field configuration A_μ^n belonging to the n class so that it satisfies, at infinity,

$$\lim_{\rho\to\infty}A_\mu^n=-\frac{i}{e}U_n^\dagger(\phi)\partial_\phi U_n(\phi)\partial_\mu\phi=\frac{1}{e}M_n\partial_\mu\phi. \tag{5}$$

One has, explicitly,

$$M_n=(n+Nk)\,\mathrm{diag}\left(\frac{1}{N},\frac{1}{N},\dots,\frac{1}{N},\frac{1-N}{N}\right) \tag{6}$$

and hence M_n can be written in terms of the $(N-1)$ $SU(N)$ generators H_i spanning the Cartan subalgebra of $SU(N)$,

$$M_n=(n+Nk)\sum_{i=1}^{N-1}m^iH_i \tag{7}$$

where m^i are the magnetic weights, as defined in [15].

In view of Eq. (5), the natural ansatz for a vortex solution with topological charge n is

$$A_\mu^n=\frac{1}{e}\partial_\mu\phi\,M_n\,a(\rho) \tag{8}$$

with $a(\rho)$ such that $G_{\mu\nu}\to 0$ as $\rho\to\infty$, fast enough to ensure the finiteness of the energy.

The finiteness of energy also requires that, at infinity, the Higgs scalars $\Phi^{(A)}$ ($A=1,2,\dots,N$) take their vacuum value, minimizing the symmetry breaking potential. Moreover,

$$\lim_{\rho\to\infty}D_\mu[\Phi^{(A)}]=0. \tag{9}$$

Condition (9) can be achieved either by taking the scalars in the Cartan algebra of $SU(N)$ or in its complement. Let us write the $SU(N)$ generators in the Cartan-Weyl basis, with H_i the $N-1$ generators spanning the Cartan algebra and $E_{\pm\alpha}$ those in its complement,

$$[H_i,E_{\pm\alpha}]=\pm\alpha_iE_{\pm\alpha}$$

$$[E_\alpha,E_{-\alpha}]=\sum_{i=1}^{N-1}\alpha_iH_i \tag{10}$$

where $\alpha_i=\alpha^i$ are the roots in an orthonormal basis. Then, one can choose the symmetry breaking potential so that the first S scalars $\bar{\Phi}^{(1)},\bar{\Phi}^{(2)},\dots,\bar{\Phi}^{(S)}$ are in the Cartan algebra and the rest, $\Phi^{(1)},\Phi^{(2)},\dots,\Phi^{(T)}$ in its complement, $S+T=N$. Now, in order to satisfy Eq. (9), one necessarily has

$$\lim_{\rho\to\infty}\bar{\Phi}^{(A)}(\rho,\phi)=\sum_{j=1}^{N-1}C_j^{(A)}H_j$$

$$\lim_{\rho\to\infty}\Phi^{(A)}(\rho,\phi)=U_n^\dagger(\phi)\left(\sum_{\pm\alpha}\eta_\alpha^{(A)}E_\alpha\right)U_n(\phi)=U_n^\dagger(\phi)\eta^{(A)}U_n(\phi) \tag{11}$$

with $C_j^{(A)}$ and $\eta_\alpha^{(A)}$ constants. The constants $\eta^{(A)}$ should be adjusted to so that they would minimize $V(\Phi^{(A)})$.

In view of the conditions described above, a consistent ansatz for a Z_N vortex configuration can be proposed in the form

$$\bar{\Phi}^{(A)}=\sum_{j=1}^{N-1}C_j^{(A)}H_j$$

$$\Phi^{(A)}=U_n^\dagger(\phi)\left(\sum_{\pm\alpha}\eta_\alpha^{(A)}\psi_\alpha^{(A)}(\rho)E_\alpha\right)U_n(\phi)$$

$$A_\phi=\frac{1}{e}a(\rho)M_n$$

$$A_\rho=A_0=A_z=0. \tag{12}$$

Here we have taken the $\bar{\Phi}^{(A)}$ scalars to be constant everywhere. $\psi_\alpha^{(A)}(\rho)$ and $a(\rho)$ should satisfy the boundary conditions

$$\lim_{\rho\to\infty}\psi_\alpha^{(A)}(\rho)=1,\qquad \lim_{\rho\to\infty}a(\rho)=n. \tag{13}$$

Ansatz (12) implies that

$$D_\phi\Phi^{(A)}=[n-a(\rho)]\partial_\phi\Phi^{(A)}. \tag{14}$$

Given the ansatz for the $\bar{\Phi}$-type scalars, the equations of motion derived from Eq. (1) take the form

$$D_\mu G^{\mu\nu}=ie\sum_{A=1}^{N-1}[D_\nu\Phi^{(A)},\Phi^{(A)}]$$

$$D_\mu D^\mu\Phi^{(A)}=\frac{\delta V}{\delta\Phi^{(A)}}. \tag{15}$$

That is, apart from the potential, the $\bar{\Phi}^{(A)}$ fields play no role in the dynamics. Concerning the other scalars $\Phi^{(A)}$, separability of the equations of motion into radial and angular parts imposes [11]

$$[M_n,[M_n,\Phi^{(A)}]]=R_n^A(\rho)\Phi^{(A)}$$

$$\sum_{A=1}^{N-1}[\Phi^{(A)},[\Phi^{(A)},M_n]]=S_n^A(\rho)M_n. \tag{16}$$

One can see that these conditions simplify the ansatz (12) to

$$\bar{\Phi}^{(A)}=\sum_{j=1}^{N-1}C_j^{(A)}H_j$$

$$\Phi^{(A)}=\eta^{(A)}\psi^{(A)}(\rho)U_n^\dagger(\phi)(E_{\alpha_A}+E_{-\alpha_A})U_n(\phi)$$

$$A_\phi=\frac{1}{e}a(\rho)M_n$$

$$A_\rho=A_0=A_z=0. \tag{17}$$

In order to characterize the vortex solutions from the topological point of view one can introduce an ‘‘electromagnetic tensor’’ $\mathcal{G}_{\mu\nu}$ analogous to that proposed by Polyakov for the $SO(3)$ monopole solution [9]. In view of the ansatz for the gauge field, it is natural to take

$$\mathcal{G}_{\mu\nu}=\frac{\mathrm{tr}(M_nG_{\mu\nu})}{[\mathrm{tr}(M_n^2)]^{1/2}}. \tag{18}$$

Then, the flux Φ associated to the magnetic field $\mathcal{G}_{12}$ reads, for the n-vortex solution,

$$\Phi=(n+Nk)\Phi_0 \tag{19}$$

with $\Phi_0=2\pi/e$. Let us recall that $n=1,2,\ldots,N-1$ indicates the topological class to which the configuration belongs while $k\in Z$ is related to gauge transformations that, although leading to the same behavior at infinity (and hence are topologically trivial), cannot be well defined everywhere and then are not gauge equivalent everywhere, thus giving, for a fixed n, different values for the magnetic flux [12].

Although the analysis of the radial equations of motion and their solution can be performed for arbitrary N, let us concentrate in the $SU(3)$ vortex solution, for which two topologically inequivalent classes exist. The associated $U_n(\phi)$ are (we take for simplicity $k=0$)

$$U_n(\phi)=e^{in\phi\lambda_8/\sqrt{3}},\quad n=1,2. \tag{20}$$

One then has

$$M_n=\frac{n\lambda_8}{\sqrt{3}}. \tag{21}$$

An explicit realization of the Cartan algebra is

$$H_1=\frac{\lambda_3}{2},\quad H_2=\frac{\lambda_8}{2} \tag{22}$$

where λ_3 and λ_8 are the diagonal Gell-Mann matrices. One then has, for the two-component magnetic weight (7),

$$\vec{m}=(0,2/\sqrt{3}). \tag{23}$$

Concerning the step generators E_α, they can be written in terms of the Gell-Mann matrices λ_i in the form

$$E_{\alpha_1}+E_{-\alpha_1}=\frac{1}{\sqrt{2}}\lambda_4$$

$$E_{\alpha_2}+E_{-\alpha_2}=\frac{1}{\sqrt{2}}\lambda_6$$

$$E_{\alpha_3}+E_{-\alpha_3}=\frac{1}{\sqrt{2}}\lambda_1. \tag{24}$$

The solution found in [7] corresponds to just one $\bar{\Phi}$-type scalar,

$$\bar{\Phi}=B\lambda_3+C\lambda_8 \tag{25}$$

and two Φ-type ones

$$\Phi^{(1)}=\frac{1}{\sqrt{6}}\eta^{(1)}\psi^{(1)}(\rho)U_n^\dagger(\phi)\lambda_4U_n(\phi)$$

$$\Phi^{(2)}=\frac{1}{\sqrt{6}}\eta^{(2)}\psi^{(2)}(\rho)U_n^\dagger(\phi)\lambda_6U_n(\phi). \tag{26}$$

With this choice, the radial equations of motion read

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\psi^{(A)}}{d\rho}\right)-\left(\frac{n-a(\rho)}{\rho}\right)^2\psi^{(A)}-v^{(A)}(\rho)\psi^{(A)}(\rho)=0$$

$$\rho\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{da}{d\rho}\right)-\frac{e}{2}[(\eta^{(1)}\psi^{(1)})^2+(\eta^{(2)}\psi^{(2)})^2][n-a(\rho)]=0 \tag{27}$$

where $v^{(A)}(\rho)$ stands for the derivative of the potential with respect to $\Phi^{(A)}$.

The symmetry breaking potential proposed in [7] can be written in the form

$$V(\Phi^{(A)},\bar{\Phi})=\frac{g_1\eta^4}{8}\sum_{A=1}^{2}\left(\frac{1}{2}\mathrm{Tr}[\Phi^{(A)}]^2-1\right)^2+\frac{\bar{g}_1\eta^4}{8}\left(\frac{1}{2}\mathrm{Tr}[\bar{\Phi}\bar{\Phi}]-1\right)^2+\frac{g_2\eta^4}{4}(\mathrm{Tr}[\Phi^{(1)}\Phi^{(2)}])^2+d_{abc}\bar{\Phi}^a\left(\sum_{A=1}^{2}f_A\Phi^{(A)\,b}\Phi^{(A)\,c}+h\Phi^{(1)\,b}\Phi^{(2)\,c}\right) \quad (28)$$

where $\Phi^{(A)}=\Phi^{(A)\,b}\lambda^b$ and d_{abc} is the completely symmetric $SU(3)$ tensor. In Eq. (28) and in our subsequent discussion we will use Higgs fields that become normalized in the limit $\rho\to\infty$. One can see that the choice of the same coupling constant g_1 for the quartic coupling of the $\Phi^{(A)}$ fields implies that $f^{(1)}=f^{(2)}$ and reduces system (27) to that arising in the $U(1)$ case, which is solved, at critical coupling, by the solutions of the original Bogomol'nyi equations.

III. TWO-HIGGS-FIELD VORTEX IN $SU(3)$ GAUGE THEORY

In this section we shall present a ''minimal'' $SU(3)$ solution with only two Higgs fields $\Phi^{(A)}$ $(A=1,2)$ in the adjoint. The Hamiltonian of the model is defined uniquely up to the Higgs potential. There is a considerable freedom in the Higgs potential. In a way we consider a Higgs potential simpler than that of the previous section, but in an other way we generalize it such that it will disallow solutions of the form discussed in Ref. [7]. Vortex solutions for a similar generalization of the $SU(2)$ Higgs potential were shown to exist in Ref. [3]

The Higgs potential we propose is identical in form to that of Ref. [3] for $SU(2)$:

$$V(\Phi^{(1)},\Phi^{(2)})=\frac{g_1\eta^4}{8}\sum_{A=1}^{2}\left(\frac{1}{2}\mathrm{Tr}[\Phi^{(A)}]^2-1\right)^2+\frac{g_2\eta^4}{4}\left(\frac{1}{2}\mathrm{Tr}[\Phi^{(1)}\Phi^{(2)}]-c\right)^2. \quad (29)$$

The generalization compared to Ref. [7] appears in the nonzero value of the constant c, that is the cosine of the ''angle'' between the two Higgs fields at infinity. The brackets of Eq. (29) must vanish at infinity to keep the Hamiltonian finite. Thus, unlike in previous models the Higgs fields are *required not to be orthogonal* at infinity. Admittedly, the model we study here is less general in the sense that the self-coupling of the two Higgs bosons is assumed to be identical.

The field equations derived from the Lagrangian, analogous to Eqs. (15) and (14), are

$$D_\mu G_{\mu\nu}-ie\eta^2\sum_{A=1}^{2}[\Phi^{(A)},D_\nu\Phi^{(A)}]=0 \quad (30)$$

and

$$D_\mu D_\mu\Phi^{(A)}-g_1\eta^2\Phi^{(A)}\left(\frac{1}{2}\mathrm{Tr}[\Phi^{(A)}]^2-1\right)-g_2\eta^2\Phi^{(B)}\left(\frac{1}{2}\mathrm{Tr}[\Phi^{(1)}\Phi^{(2)}]-c\right)=0, \quad (31)$$

where $A=1,2$ and then $B=2,1$.

As in the previous section, the ansatz we use for finding vortex solutions is based on the philosophy that the vortex solution is associated with a singular gauge transformation that maps circles linked with the vortex to a smooth transformation connecting two elements of the center. Choosing $U_n(\phi)$, as in Eq. (20), the Higgs fields are defined as

$$\Phi^{(i)}(x)=U_n(\phi)\psi^{(i)}(\rho)U_n^\dagger(\phi), \quad (32)$$

where $i=1,2$ for the two Higgs bosons.

The ansatz for the gauge field,

$$A_\mu(x)=\partial_\mu\phi[a_8(\rho)\lambda_8+a_3(\rho)\lambda_3], \quad (33)$$

is diagonal in gauge space. We will later show that unlike for vortices of the previous section the component $a_3(\rho)$ must be different from zero, despite the fact that this component does not contribute to the vortex at $\rho\to\infty$. The gauge field of Eq. (33) satisfies the gauge fixing condition $\partial_\mu A_\mu=0$. Taking the derivative of the Higgs field generates a vortex contribution in the λ_8 gauge direction. The form of the gauge field was chosen to be able to cancel this vortex at infinity in the covariant derivative. Without such a cancellation the term of the Hamiltonian containing the covariant derivative of the Higgs fields would diverge.

We still need to show that the forms chosen for the fields are consistent with field equations (30) and (31). Before doing so we will further restrict the form of our solution. We will assume that the Higgs fields have only components

$$\psi^{(A)}=\psi_4^{(A)}\lambda_4+\psi_6^{(A)}\lambda_6, \quad (34)$$

where λ_4 and λ_6 are off-diagonal Gell-Mann matrices matrices. Two is the minimal number of components needed to satisfy the all the constraints on the normalization of the Higgs fields at $\rho\to\infty$ simultaneously. The two Higgs fields, provided their coefficients are not identical, break $SU(3)$ symmetry completely, down to its center, Z_3.

Let us now show that the gauge structure we propose is consistent with the field equations. First of all consider Eq. (31). The two equations, for the choices $A=1$ and 2, are consistent with the solution $\psi_I^{(1)}=\pm\psi_I^{(2)}$. We will show that the choice

$$\psi_4^{(1)}=\psi_4^{(2)}\equiv\psi_4,\quad \psi_6^{(1)}=-\psi_6^{(2)}\equiv\psi_6 \quad (35)$$

is also consistent with Eq. (30). Under the assumptions (32)–(35), Eq. (30) can be calculated as

$$D_\mu G_{\mu\nu} - ie\eta^2 \sum_{A=1}^{2} [\Phi^{(A)}, D_\nu \Phi^{(A)}] = \partial_\mu \phi \left[\lambda_8 \rho \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{da_8}{d\rho}\right) + \lambda_3 \rho \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{da_3}{d\rho}\right)\right] - 2e\,\eta^2 \partial_\mu \phi [(\psi_4)^2 e\, a_+ (\sqrt{3}\lambda_8 + \lambda_3) + (\psi_6)^2 \times e\, a_- (\sqrt{3}\lambda_8 - \lambda_3)] = 0, \tag{36}$$

where

$$a_\pm = \sqrt{3} a_8 + \frac{n}{e} \pm a_3. \tag{37}$$

Clearly the space and isospace structures are consistent and Eq. (36) leads to two scalar equations for the two unknown functions, a_+ and a_-. These equations are

$$\rho \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{da_+}{d\rho}\right) - 4e^2\eta^2 [2(\psi_4)^2 a_+ + (\psi_6)^2 a_-] = 0 \tag{38}$$

and

$$\rho \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{da_-}{d\rho}\right) - 4e^2\eta^2 [(\psi_4)^2 a_+ + 2(\psi_6)^2 a_-] = 0. \tag{39}$$

In a similar way, the scalar equations reduce to two equations for the two components, ψ_4 and ψ_6:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\psi_4}{d\rho}\right) - \frac{a_+^2}{\rho^2}\psi_4 - g_1 \eta^2 \psi_4 (\psi_4^2 + \psi_6^2 - 1) - g_2 \eta^2 \psi_4 (\psi_4^2 - \psi_6^2 - c) = 0 \tag{40}$$

and

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\psi_6}{d\rho}\right) - \frac{a_-^2}{\rho^2}\psi_6 - g_1 \eta^2 \psi_6 (\psi_4^2 + \psi_6^2 - 1) + g_2 \eta^2 \psi_6 (\psi_4^2 - \psi_6^2 - c) = 0. \tag{41}$$

The boundary conditions for the four fields are the following:

$$a_\pm(0) = \frac{n}{e}$$

$$\lim_{\rho\to\infty} a_\pm(\rho) = 0$$

$$\psi_4(0) = \psi_6(0) = 0$$

$$\lim_{\rho\to\infty} \psi_4(\rho) = \sqrt{\frac{1+c}{2}}, \quad \lim_{\rho\to\infty} \psi_6(\rho) = \sqrt{\frac{1-c}{2}}. \tag{42}$$

Now at this point it should be obvious that $a_3 = 0$, equivalent to $a_+ = a_-$ is not an admissible solution. If $a_+ = a_-$, then from Eqs. (38) and (39) it follows that $\psi_4 = \psi_6$. Such a solution would not satisfy the boundary condition (42), unless $c = 0$.

Note that at $c=0$ $\psi_4 = \psi_6$ and $a_+ = a_-$. In other words the a_3 component of the gauge field vanishes. Then, after appropriate rescaling, the vortex defined by Eqs. (38)–(41) coincides with that defined by Eq. (27), provided we choose $g_1 = \overline{g_1}$ and $\eta^{(1)} = \eta^{(2)}$.

We have not been able to prove analytically the existence of solutions of these equations. In a future publication [16] we will study the solutions numerically. At special values of the couplings, however, the second order equations become first order. The system of equations also decouples and can be rescaled to a form identical to a combination of two critical Abelian vortices. Abelian vortices have been well studied [11] and the existence of solutions has been shown.

The form of the solutions for two-adjoint-Higgs-boson model is unique up to gauge transformations. A gauge transformation can always bring $U_n(\phi)$ to the form used above. Then the gauge field, commuting with $U_n(\phi)$, should only have components a_8 or a_3. Furthermore, the combination of constraint

$$\sum_A [\Phi^{(A)}, \partial_\mu \Phi_{(A)}] = 0$$

and of the field equations for the two-Higgs-boson fields can only be satisfied with at most two nonvanishing components of $\Phi^{(A)}$. Choosing these as Φ_4 and Φ_6 we arrive at the choice of this section.[1]

IV. CRITICAL COUPLING

At a critical coupling the second order differential equations for the gauge and Higgs field of Abelian vortex solutions can be transformed to linear equations [11,10]. We will show below that the solution found in the previous section also satisfies linear equations at a critical coupling.

First of all, it will be advantageous to express Hamiltonian (1) in terms of the Higgs component a_8, a_3 (or a_+ and a_-), ψ_4, and ψ_6. One obtains

[1]Components that can be transformed into each other by global $U(1)\times U(1)$ transformations and therefore satisfy the same field equations are not counted as different. For our choice of the components Φ_5 and Φ_7 can be eliminated by global $U(1)\times U(1)$ transformations.

$$H=2\pi\int_0^\infty \rho\, d\rho\left[\frac{1}{2\rho^2}(a_8')^2+\frac{1}{2\rho^2}(a_3')^2+\frac{\eta^2 e^2}{\rho^2}(\psi_4^2 a_+^2+\psi_6^2 a_-^2)+\eta^2(\psi_4'^2+\psi_6'^2)+\frac{g_1\eta^4}{4}(\psi_4^2+\psi_6^2-1)^2+\frac{g_2\eta^4}{4}(\psi_4^2-\psi_6^2-c)^2\right]. \tag{43}$$

The variation of (43) results in field equations (38)–(41).

Inspired by Ref. [12] we rearrange the Hamiltonian into an alternative form

$$H=2\pi\int_0^\infty \rho\, d\rho\left\{\eta^2\left[\psi_4'+\frac{\gamma e}{\rho}\psi_4 a_+\right]^2+\eta^2\left[\psi_6'+\frac{\delta e}{\rho}\psi_6 a_-\right]^2+\frac{1}{2\rho^2}[a_8'-\alpha\rho\eta^2(\psi_4^2+\psi_6^2-1)]^2+\frac{1}{2\rho^2}[a_3'-\beta\rho\eta^2(\psi_4^2-\psi_6^2-c)]^2+\frac{f_1\,\eta^4}{4}(\psi_4^2+\psi_6^2-1)^2+\frac{f_2\,\eta^4}{4}(\psi_4^2-\psi_6^2-c)^2+\frac{1}{\rho}\frac{dX}{d\rho}\right\}, \tag{44}$$

where α, β, γ, and δ are yet undetermined constants and X is an undetermined form. Comparing Eq. (44) with Eq. (43) provides the following values for the constants:

$$\gamma=\delta=\frac{n}{|n|} \tag{45}$$

$$\alpha=\sqrt{3}\beta=-\sqrt{3}e\frac{n}{|n|}. \tag{46}$$

Furthermore, one obtains

$$X=\frac{n}{|n|}\eta^2 e\left[\left(\psi_4^2-\frac{1+c}{2}\right)a_+ +\left(\psi_6^2-\frac{1-c}{2}\right)a_-\right], \tag{47}$$

$$f_1=g_1-24e^2, \tag{48}$$

and

$$f_2=g_2-8e^2. \tag{49}$$

Substituting these values back into the Hamiltonian we can see that the Hamiltonian is minimized with a minimum value of $2\pi|n|$ at the critical couplings

$$g_1=24e^2 \tag{50}$$

and

$$g_2=8e^2 \tag{51}$$

if (for positive n) the fields satisfy the following Bogomol'nyi-type equations in terms of the original fields, a_8 and a_3:

$$\psi_4'=-\frac{e}{\rho}\psi_4\left(\sqrt{3}a_8+\frac{n}{e}+a_3\right), \tag{52}$$

$$\psi_6'=-\frac{e}{\rho}\psi_6\left(\sqrt{3}a_8+\frac{n}{e}-a_3\right), \tag{53}$$

$$a_8'=-\eta^2\rho\sqrt{3}e(\psi_4^2+\psi_6^2-1), \tag{54}$$

and

$$a_3'=-\eta^2\rho e(\psi_4^2-\psi_6^2-c). \tag{55}$$

The value of the Hamiltonian at the critical point is equal to

$$H=2\pi[X(\infty)-X(0)]=2\pi|n|\eta^2, \tag{56}$$

proportional to the topological charge.

V. CONCLUSIONS

A new vortex solution was shown to exist in $SU(3)$ gauge theory with two adjoint Higgs bosons. This can be contrasted with the solution found in Ref. [7], which requires three adjoint Higgs bosons. At a critical value of the Higgs self-coupling (where the gauge and Higgs boson masses coincide) the Hamiltonian has an exact lower bound and the Higgs and gauge fields satisfy first order Bogomol'nyi type field equations. In previous solutions such a phenomenon signaled an increase of the underlying supersymmetry [13,14]. We expect that the situation is similar in the case discussed in this paper. The non-Abelian vortices discussed here could play a relevant role in the confinement scenario arising in strongly coupled supersymmetric theories [17].

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